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SOME APPLICATIONS OF HENKIN QUANTIFIERS[†]

BY

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ABSTRACT

We show how to approximate a Henkin formula by first order formulas. This method of approximation is then applied to problems of axiomatizing classes of structures.

1. Introduction

Let L be a first order language with a finite[§] number of nonlogical symbols and let $\mathscr{K}$ be a class of L-structures. $\mathscr{K}$ is *compact* if, for any set S of sentences of L, if every finite subset of S has a model in $\mathscr{K}$ then S itself has a model in $\mathscr{K}$. A set T of sentences of L is a set of axioms for (the first ordered properties of) $\mathscr{K}$ if, for all sentences ϕ of L, ϕ is true in all $\mathfrak{M} \in \mathscr{K}$ iff $T \vdash \phi$. Notice that the following are obviously equivalent:

a) $\mathscr{K}$ is compact and T is a set of axioms for $\mathscr{K}$;

b) A set S of sentences has a model in $\mathscr{K}$ iff $T \cup S$ is consistent;

c) $\mathscr{K}$ is compact and, for any L-structure $\mathfrak{M}$, $\mathfrak{M}$ is a model of T iff there is an $\mathfrak{N} \in \mathscr{K}$ with $\mathfrak{M} \equiv \mathfrak{N}$.

A class $\mathscr{K}$ is *nearly axiomatizable* if it is compact and has a recursive set of axioms for its first order properties. Less precisely, but more dramatically, $\mathscr{K}$ is nearly axiomatizable if there is a recursive set T of axioms of L which can be added to the standard first order axioms so that the Gödel Completeness and Compactness Theorems hold when "model" is taken to mean "model in $\mathscr{K}$".

In this paper we describe a simple method, which we call "straightening out Henkin quantifiers", for finding explicit axioms for classes $\mathscr{K}$ which, on some grounds, can be seen to be nearly axiomatizable. Historically, the following fact

[†] Dedicated to the memory of Abraham Robinson.

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§ The results of this paper extend to countable languages. See Remark 2.6 in [2].

(1.1) has made it much easier to see that a given class $\mathcal{K}$ is nearly axiomatizable than to actually find an explicit set of near axioms.

LEMMA 1.1. *Let $\mathcal{K}$ be a class of L-structures, $\mathcal{K}$ is nearly axiomatizable if there is a recursive expansion L' of L and a recursive set T' of L'-sentences with the following property: for all sets S of L-sentences, S has a model in $\mathcal{K}$ iff $S \cup T'$ is consistent.*

PROOF. $\mathcal{K}$ is clearly compact. The set of theorems of T' in the language L is R.E., by the completeness theorem, so has a recursive set T of axioms, by the well known result of Craig [5]. □

We developed the method discussed in this paper in the course of working out a special case (Open problem 11, p. 513, in Chang–Keisler [3]): find an explicit set of axioms for the class $\mathcal{K}$ of models $\langle M, U, \cdots \rangle$ where $\mathrm{Card}(M) \geqq \beth_\omega(\mathrm{Card}(U))$. The theorem of Vaught [18], with 1.1, makes it clear that this $\mathcal{K}$ is nearly axiomatizable. Our solution to this problem, and some refinements of Vaught's Theorem which are consequences of it, appear in [1]. A different solution was obtained independently, and a few weeks earlier, by J. Schmerl [16].

2. How to straighten out a Henkin quantifier

A Henkin formula $H(\mathbf{z})$ (the boldface $\mathbf{z}$ indicates a finite sequence $z_1 \cdots z_k$ of variables) is an expression of the form

$$\left.\begin{matrix} \forall \mathbf{x}_1 \exists y_1 \\ \forall \mathbf{x}_2 \exists y_2 \\ \cdot \\ \cdot \\ \cdot \\ \forall \mathbf{x}_n \exists y_n \end{matrix}\right\} \phi(\mathbf{z}, \mathbf{x}_1, y_1, \cdots, \mathbf{x}_n, y_n),$$

where ϕ is an ordinary first order formula. The formula $H(\mathbf{z})$ is read "for all $\mathbf{x}_1, \cdots, \mathbf{x}_n$ there exist $y_1, \cdots, y_n$ with y_i depending only on the sequence $\mathbf{x}_i$ such that $\phi(\mathbf{z}, \mathbf{x}_1, y_1, \cdots, \mathbf{x}_n, y_n)$." Thus, the semantics of Henkin formulas is given by: $\mathfrak{M} \models H(\mathbf{z})$ iff there are functions $G_1, \cdots, G_n$ (of the appropriate number of arguments) such that $(\mathfrak{M}, G_1 \cdots G_n)$ is a model of $\forall \mathbf{x}_1 \cdots \forall \mathbf{x}_n$ $\phi(\mathbf{z}, \mathbf{x}_1, G_1(\mathbf{x}_1), \cdots, \mathbf{x}_n, G_n(\mathbf{x}_n))$.

Henkin quantifier prefixes were introduced in Henkin [9] and studied in Enderton [7] and Walkoe [19]. The importance of Henkin formulas for our purposes is that:

a) any recursive T' as in Lemma 1.1 can, theoretically at least, be expressed by a Henkin sentence of L;

b) it is easy to axiomatize the first order consequences of a Henkin formula.

Contention (a) is justified by the following two theorems.

THEOREM 2.1. (Kleene [13]). *Let L' be a recursive expansion of L and let T' be a recursive theory of L'. There is a single second order Σ_1^1 sentence $\exists R\psi(R)$ of L such that, for any infinite L-structure $\mathfrak{M}$, $\mathfrak{M}$ is a model of $\exists R\psi(R)$ if and only if some expansion $\mathfrak{M}' = (\mathfrak{M}, \cdots)$ of $\mathfrak{M}$ is a model of T'.*

Having now defined Σ_1^1 formula, we can observe that every Henkin formula is equivalent to a Σ_1^1 formula. Rather surprisingly, the converse is also true.

THEOREM 2.2. (Enderton [7], Walkoe [19]). *Every Σ_1^1 formula $\exists R\psi(R, z)$ of L is equivalent to a Henkin formula $H(z)$ of L.*

We won't actually use the above theorems. They are included for moral support in our search for Henkin formulas expressing theorems T' as in 1.1. Actually, we seldom express the entire T' by a single Henkin sentence, but rather use sets of Henkin formulas.

MAIN DEFINITION. Given a Henkin formula $H(z) =$

$$\left.\begin{matrix} \forall x_1 \exists y_1 \\ \cdot \\ \cdot \\ \cdot \\ \forall x_n \exists y_n \end{matrix}\right\} \phi(z, x_1, y_1, \cdots, x_n, y_n)$$

as above, we define its *first-order approximations* as follows. First some notation to make the formulas more readable. We use $\mathbf{y}$ for $y_1, \cdots, y_n$ and $\vec{x}$ for $x_1, \cdots, x_n$. We add new superscripts to variables freely, writing $\mathbf{y}^1$ for $y_1^1, \cdots, y_n^1$ and similarly for $\mathbf{y}^2$, etc. We write $\vec{x}^1$ for the sequence $x_1^1, \cdots, x_n^1$ and similarly for $\vec{x}^2$, etc. We write $\mathbf{y}^1 = \mathbf{y}^2$ for the conjunction $(y_1^1 = y_1^2 \wedge \cdots \wedge y_n^1 = y_n^2)$. We think of $\mathbf{y}^1$ and $\mathbf{y}^2$ as first and second choices for the sequence $\mathbf{y}$.

First approximation:

$$\forall x_1^1 \cdots \forall x_n^1 \exists y_1^1 \cdots \exists y_n^1 \phi(z, x_1^1, y_1^1, \cdots, x_n^1, y_n^1)$$

or, more simply, using the above notation,

$$\forall \vec{x}^1 \exists y^1 \phi(z, \vec{x}^1, y^1).$$

Second approximation:

$$\forall \vec{x}^1 \exists y^1 \forall \vec{x}^2 \exists y^2 [\phi(z, \vec{x}^1, y^1) \wedge \phi(z, \vec{x}^2, y^2) \wedge \bigwedge_{1 \le i \le n} (x_i^1 = x_i^2 \rightarrow y_i^1 = y_i^2)].$$

k-th approximation:

$$\forall \vec{x}^1 \exists y^1 \cdots \forall \vec{x}^k \exists y^k [\bigwedge \phi(z, \vec{x}^j, y^j) \wedge \bigwedge (x_i^j = x_i^{j'} \rightarrow y_i^j = y_i^{j'})],$$

where the first big conjunction ranges over j between 1 and k $(1 \leqq j \leqq k)$ and the second ranges over i between 1 and n, and over j and j' between 1 and k.

We collect some obvious facts in the next lemma. We will use these pretty much without comment later on.

LEMMA 2.3.

i) *The* $(k+1)$*-th approximation to* $H(z)$ *logically implies the* k*-th approximation.*

ii) *The Henkin formula* $H(z)$ *logically implies each of its approximations.*

iii) *The set of approximations to* $H(z)$ *is (primitive) recursive.*

PROOF. Obvious. □

The converse of 2.3 (ii) is not valid except on resplendent structures. We refer the reader to Barwise–Schlipf [2] for the few simple facts we use about resplendent[†] models. The notation $\mathfrak{M} <_{\Sigma_1^1} \mathfrak{N}$ indicates that $\mathfrak{M} < \mathfrak{N}$ and that for any Σ_1^1 formula with parameters from $\mathfrak{M}$ which holds in $\mathfrak{N}$ also holds in $\mathfrak{M}$.

THEOREM 2.4. *Let* $\mathfrak{M}$ *be a structure for L. The following are equivalent*[‡]:

i) $\mathfrak{M}$ *is resplendent; i.e., for all* $\mathfrak{N}$, $\mathfrak{M} < \mathfrak{N}$ *implies* $\mathfrak{M} <_{\Sigma_1^1} \mathfrak{N}$.

ii) *For each Henkin formula* $H(z)$, $\mathfrak{M}$ *is a model of*

$$\forall z [H(z) \leftrightarrow \bigwedge \mathrm{App}_H(z)],$$

where $\mathrm{App}_H(z)$ *is the set of first order approximations to* $H(z)$.

PROOF. The implication (ii) ⇒ (i) is immediate by Theorem 2.2. To prove

[†] All saturated and special models are resplendent (see theor. 5.3.1 and ex. 5.3.5 in Chang–Keisler [3]) so the reader is welcome to use these notions in place of resplendence below. This is really making life harder, however, since it is easy to see that resplendent models exist in all powers. See [2].

[‡] The condition given in (i) is not the correct definition of resplendence if L is countable infinite. The implication (ii) → (i) is the only result of this paper which does not hold for countable languages.

(i) $\Rightarrow$ (ii), we use the Π_1^1 reflection principle for resplendent models, due to Schlipf [15] and found as 2.4 (vi) of [2]. This states that any Π_1^1 formula (the dual of a Σ_1^1 formula) which holds in $\mathfrak{M}$ of some z also holds in some (actually all) *countable*, recursively saturated elementary submodels of $\mathfrak{M}$. Thus it suffices to assume that $\mathfrak{M}$ is countable, recursively saturated, satisfies $\mathrm{App}_H(z)$ and prove that $\mathfrak{M} \models H(z)$.We simplify the notation in the proof by considering a simple but typical case, namely where $H(z)$ is:

$$\left.\begin{matrix} \forall x_1 \exists y_1 \\ \\ \forall x_2 \exists y_2 \end{matrix}\right\} \phi(z, x_1, y_1, x_2, y_2).$$

The first couple of approximations, written out in detail, are:

First approximation: $\forall x_1^1 x_2^1 \exists y_1^1 y_2^1 \phi(z, x_1^1, y_1^1, x_2^1, y_2^1)$

Second approximation: $\forall x_1^1 x_2^1 \exists y_1^1 y_2^1 \forall x_1^2 x_2^2 \exists y_1^2 y_2^2\ \phi(z, x_1^1, y_1^1, x_2^1, y_2^1) \wedge$

$$\phi(z, x_1^2, y_1^2, x_2^2, y_2^2) \wedge (x_1^1 = x_1^2 \rightarrow y_1^1 = y_1^2) \wedge (x_2^1 = x_2^2 \rightarrow y_2^1 = y_2^2).$$

Let $\psi_k^0(z)$ be the k-th approximation. Let $\psi_k^1(z, x_1^1, y_1^1, x_1^2, y_2^2)$ be the result of stripping the first four quantifiers $\forall x_1^1 x_2^1 \exists y_1^1 y_2^1$ from $\psi_k^0(z)$. In fact, let $\psi_k^{i+1}(z, x_1^1, \cdots, y_2^{i+1})$ be the result of stripping the first four quantifiers off ψ_k^i, at least if $i + 1 \leqq k$. The formula ψ_k^k has had all the quantifiers (except those hidden in ϕ) stripped off so we let ψ_k^i be ψ_k^k for $i \geqq k$. Notice that ψ_k^i and $\psi_{k'}^i$ have the same free variables and that if $k' \geqq$ then $\psi_{k'}^i$ logically implies ψ_k^i. Given all this notation, we can begin the proof in earnest. Let

$$(*) \qquad \langle a_1^1, a_2^1 \rangle, \langle a_1^2, a_2^2 \rangle, \cdots, \langle a_1^k, a_2^k \rangle, \cdots$$

be an enumeration of the countable set $M \times M$. We want to define a sequence

$$(**) \qquad \langle b_1^1, b_2^1 \rangle, \langle b_1^2, b_2^2 \rangle, \cdots, \langle b_1^k, b_2^k \rangle, \cdots$$

such that, for each i,

$$(1) \qquad \mathfrak{M} \models \phi(z, a_1^i, b_1^i, a_2^i, b_2^i)$$

and such that the expressions

$$G_1(a_1^i) = b_1^i, \quad \text{and} \quad G_2(a_2^i) = b_2^i$$

define functions G_1, G_2 from M into M, which will show that $\mathfrak{M} \models H(z)$. To make sure that the above expressions make G_1 and G_2 well-defined, we need to satisfy the following conditions as we define the sequence (**):

(2) $a_1^i = a_1^{i'}$ implies $b_1^i = b_1^{i'}$, and

(3) $a_2^i = a_2^{i'}$ implies $b_2^i = b_2^{i'}$.

The first order approximations of $H(z)$ are designed to insure that (1)–(3) can be fulfilled. The sequence (**) is defined by induction. Since $\mathfrak{M} \models \mathrm{App}_H(z)$, since $\mathfrak{M}$ is recursively saturated, and since $\psi_{k'}^1$ implies ψ_k^1 for $k' \geqq k$, there is a pair $\langle b_1^1, b_2^1 \rangle$ such that $\mathfrak{M}$ is a model of

$$\psi_k^1(z, a_1^1, b_1^1, a_2^1, b_2^1),$$

for all k. By the same argument, we can find $\langle b_1^i, b_2^i \rangle$ such that $\mathfrak{M}$ satisfies

$$\psi_k^i(z, a_1^1, b_1^1, a_2^1, b_2^1, \cdots, a_1^i, b_1^i, a_2^i, b_2^i),$$

for all k. Given the meaning of ψ_k^k, we see that the sequence does indeed satisfy (1)–(3). □

The following sequence of corollaries is not directly related to the method we are discussing, but is included to show that the Craig Interpolation Theorem is a simple consequence of Theorem 2.4.

COROLLARY 2.5. *Let $\mathfrak{M}$ be a resplendent model and let $H_1(z)$ and $H_2(z)$ be Henkin formulas which define disjoint relations on $\mathfrak{M}$. There are approximations $\theta_1(z)$ and $\theta_2(z)$ of H_1 and H_2 such that θ_1 and θ_2 define disjoint relations on $\mathfrak{M}$.*

PROOF. This is immediate from 2.4. □

COROLLARY 2.7. (Craig Interpolation Theorem). *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be disjoint Σ_1^1 definable classes of L-structures. There is an elementary $\mathcal{K}$ (definable by a single sentence of L) containing $\mathcal{K}_1$ and disjoint from $\mathcal{K}_2$.*

PROOF. By 2.2 we may assume that $\mathcal{K}_i$ is defined by a Henkin sentence H_i. We claim that there is some approximation of H_1 whose class $\mathcal{K}$ of models separates $\mathcal{K}_1$ from $\mathcal{K}_2$. If not, then the set $\mathrm{App}_{H_1} \cup \mathrm{App}_{H_2}$ is consistent and hence has a resplendent model $\mathfrak{M}$. But then, by 2.4, $\mathfrak{M}$ is in both $\mathcal{K}_1$ and $\mathcal{K}_2$. □

For simple applications of the method of straightening out a Henkin quantifier (as in Section 3), Theorem 2.4 will suffice. For more complicated examples (as in [2] especially, but also in Section 6), it behoves us to prove a result which allows us to piece an infinite number of Henkin formulas together.

THEOREM 2.8. *Let $\{H_n(z_1 \cdots z_{f(n)}) \mid n < \omega\}$ be a recursive set of Henkin formulas of the form $H_n(z) =$*

$$\left.\begin{array}{c}\forall \boldsymbol{x}_1 \exists y_1 \\ \cdot \\ \cdot \\ \cdot \\ \forall \boldsymbol{x}_{g(n)} \exists y_{g(n)}\end{array}\right\} \phi_n(z, \boldsymbol{x}_1, y_1, \cdots, \boldsymbol{x}_{g(n)}, y_{g(n)}),$$

where f and g are nondecreasing recursive functions and where $(\phi_{n+1} \rightarrow \phi_n)$ is logically valid. Let $\mathfrak{M}$ be a resplendent model of each formula $\exists z H_n(z)$. Then $\mathfrak{M}$ is a model of:

$$\exists z_1 \cdots z_n \cdots \left\{\begin{array}{c}\forall \boldsymbol{x}_1 \exists y_1 \\ \cdot \\ \cdot \\ \forall \boldsymbol{x}_{g(n)} \exists y_{g(n)} \\ \cdot \\ \cdot \\ \cdot\end{array}\right\} \bigwedge{}_{n<\omega} \phi_n(z, \boldsymbol{x}_1, y_1, \cdots).$$

PROOF. Introduce a recursive expansion L' of L by adding constant symbols $c_1, c_2, \cdots$ and function symbols $G_1, G_2, \cdots$ and consider the theory T' whose axioms are all expressions of the form

$$\forall \boldsymbol{x}_1 \cdots \forall \boldsymbol{x}_{g(n)} \phi_n(c_1, \cdots, c_{f(n)}, \boldsymbol{x}_1, G_1(\boldsymbol{x}_1), \cdots, \boldsymbol{x}_{g(n)}, G_{g(n)}(\boldsymbol{x}_{g(n)})).$$

The hypothesis implies that the theory Th($\mathfrak{M}$) of $\mathfrak{M}$ is consistent with each finite subset of T'. Since $\mathfrak{M}$ is resplendent, $\mathfrak{M}$ can be expanded to a model of T', by [2], 2.4 (v). But this is exactly the meaning of the conclusion of the theorem. □

3. The first order properties of models with involutions

An *involution* is simply a nontrivial automorphism f of order 2, $f^2 =$ identity. We illustrate the method of axiomatizing nearly axiomatizable classes with the simple case where $\mathcal{K}$ is the class of all L-structures with involutions.

LEMMA 3.1. *The class of models with involutions is nearly axiomatizable.*

PROOF. This follows immediately from Lemma 1.1 since S has a model with an involution if and only if S is consistent with the following sentence involving a new symbol f:

$$\exists z(fz \neq z) \wedge \forall x, y\,[ffx = x \wedge [xEy \leftrightarrow fxEfy)].$$

We are treating the special case where L has only one binary symbol E, but an obvious modification takes care of the general case. □

Having restricted ourselves to the special case where L has only a single binary symbol in the above proof, we might as well continue to treat this special case below.

LEMMA 3.2. *A structure $\mathfrak{M}$ has an involution iff it is a model of $\exists zH(z)$, where $H(z)$ is*:

$$\left.\begin{array}{l}\forall x_0 \exists y_0 \\ \forall x_1 \exists y_1\end{array}\right\} \quad \left\{\begin{array}{l} x_0 = z \rightarrow y_0 \neq z \\ x_0 = x_1 \rightarrow y_0 = y_1 \\ x_0 E x_1 \leftrightarrow y_0 E y_1 \\ x_1 = y_0 \rightarrow y_1 = x_0. \end{array}\right.$$

PROOF. First write out the Skolem form of $H(z)$ as a formula involving two unary functions G_1 and G_2. The second conjunct in the matrix of $H(z)$ implies that $G_1 = G_2$. Rewriting it with one symbol G it says exactly that G is an involution with $G(z) \neq z$. □

All we would have to do now is to straighten out $\exists zH(z)$. On doing so, however, one notices that the theory can be made more understandable by changing bound variables, throwing in some redundant conjuncts and permuting some quantifiers.

DEFINITION. Let *Inv* be the first order theory whose axioms are all prenex sentences beginning with a quantifier string

$$\exists z \forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k$$

followed by a conjunction of formulas of the following forms:

$$(x_i = z \rightarrow y_i \neq z)$$

$$(x_i = x_j \rightarrow y_i = y_j)$$

$$(y_i = x_j \rightarrow y_j = x_i)$$

$$(x_i E x_j \leftrightarrow y_i E y_j).$$

THEOREM 3.3. *Any structure $\mathfrak{M}$ with an involution is a model of* Inv. *Any resplendent model of* Inv *has an involution.*

PROOF. Let f be an involution of $\mathfrak{M}$, $f(z) \neq z$. For any x_i pick $y_i = f(x_i)$. This clearly makes all the above conjunctions true. Now let $\mathfrak{M}$ be a resplendent model of Inv, and let $H(z)$ be as in 3.2. Consider any sentence of the form $\exists z \psi(z)$, where $\psi \in \mathrm{App}_H(z)$. This sentence has the form

$$\exists z \forall x_0 x_1 \exists y_0 y_1 \forall x_2 x_3 \exists y_2 y_3 \cdots \forall x_{2k} x_{2k+1} \exists y_{2k} y_{2k+1}$$

followed by a conjunction of formulas as above. Since $\forall x_i \exists y_i \forall x_{i+1} \exists y_{i+1}$ is stronger than $\forall x_i x_{i+1} \exists y_i y_{i+1}$, $\exists z \psi(z)$ is implied by a sentence of Inv. Thus $\mathfrak{M}$ is a model of each such $\exists z \psi(z)$. By the recursive saturation of $\mathfrak{M}$, there is a z such that $\mathfrak{M} \models \mathrm{App}_H(z)$. By Theorem 2.4, $\mathfrak{M} \models H(z)$, so $\mathfrak{M}$ has an involution, by 3.2. □

COROLLARY 3.4.[†] *The theory* Inv *is a set of axioms for the first order properties of models with involutions.*

PROOF. This is an immediate consequence of the above theorem and the fact that every model has a resplendent elementary extension, see [2]. □

It is fairly obvious that no model of ZFC, Zermelo–Fraenkel set theory with choice, has an involution. Hence ZFC$\vdash \neg \theta$, for some $\theta \in$ Inv. What is the simplest such θ? On the other hand, Cohen [4] proves that ZF does have models with involutions. Would it be any simpler to prove directly that ZF is consistent with each axiom of Inv.

Actually, in the case of ZF, Theorem 3.3 can be improved. One can show that any countable non-ω-model of ZF + Inv has an involution, even those that are not recursively saturated. (This follows directly from theor. 2.7 in the appendix of [0], plus the observation that, for ZF, the existence of an involution is a *strict*-Σ^1_1 statement, not just Σ^1_1, and hence is Π^0_1 on $\mathrm{Cov}_{\mathfrak{M}}$, for $\mathfrak{M}$ countable.) Friedman has recently obtained much stronger results on models of ZF with involutions.

There is nothing sacred about automorphisms of order 2 in all of the above. We could equally well have axiomatized the first order properties of models with automorphisms of order 33. One would just replace the third sort of conjunct above with those of the form

$$[(x_{i_{33}} = y_{i_{32}}) \wedge (x_{i_{32}} = y_{i_{31}}) \wedge \cdots \wedge (x_{i_2} = y_{i_1}) \rightarrow (y_{i_{33}} = x_{i_1})].$$

[†] Shelah informs me that he discovered this result some years ago in response to an open problem list in a preprint version of Chang–Keisler [3]. Both the problem and its solution seem to have disappeared in the final version.

4. The first order properties of homomorphisms which have splittings

Let f be a homomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are L-structures. We can think of the homomorphism f as the structure $(\mathfrak{M}, \mathfrak{N}, f)$ for a two-sorted logic L^*. We use x's (with sub and superscripts) for variables over $\mathfrak{M}$ and y's for variables over $\mathfrak{N}$.

A *splitting* for a homomorphism f from $\mathfrak{M}$ onto $\mathfrak{N}$ is an embedding g of $\mathfrak{N}$ into $\mathfrak{M}$ such that $f(g(y)) = y$, for all $y \in N$.

LEMMA 4.1. *The class of homomorphisms which have splittings is definable by a Σ_1^1 sentence and hence is nearly axiomatizable.*

In this section we axiomatize those first order properties which must hold of all homomorphisms which have splittings. Or, looked at negatively, we axiomatize those first order properties P such that "not P" can obstruct the existence of a splitting.

LEMMA 4.2. *f has a splitting iff $(\mathfrak{M}, \mathfrak{N}, f)$ is a model of the following Henkin sentence:*

$$\left.\begin{matrix} \forall y_0 \exists x_0 \\ \forall y_1 \exists x_1 \end{matrix}\right\} \left\{\begin{array}{l} y_0 = y_1 \to x_0 = x_1 \\ f(x_0) = y_0 \\ R(y_0, y_1) \to R(x_0, x_1). \end{array}\right.$$

(We are treating the case where L has only one binary R to illustrate the lemma.)

PROOF. The first line makes the two Skolem functions one, the second makes this function an inverse to f and the third makes it a homomorphism. It must of necessity be one-one. □

Now we simply straighten out the above Henkin sentence.

DEFINITION. Let *Spl* be the first order theory of L^* whose axioms are all prenex formulas beginning with a quantifier string

$$\forall y_0 \exists x_0 \forall y_1 \exists x_1 \cdots \forall y_k \exists x_k$$

followed by a conjunction of formulas of the following forms:

$$(y_i = y_j \to x_i = x_j)$$

$$(f(x_i) = y_i)$$

$$(R(y_i, y_j) \to R(x_i, x_j)).$$

A typical *consequence* of Spl is the sentence

$$\forall x_0 x_1 [R(x_0, x_1) \wedge f(x_0) = f(x_1) \rightarrow R(x_0, x_0) \vee R(x_1, x_1)].$$

THEOREM 4.3. *Let f be a homomorphism such that $(\mathfrak{M}, \mathfrak{N}, f)$ is resplendent. Then f has a splitting iff $(\mathfrak{M}, \mathfrak{N}, f) \models \mathrm{Spl}$.*

PROOF. The proof is entirely analogous to the proof of Theorem 3.3. □

COROLLARY 4.4. Spl *is a set of axioms for the first order properties of homomorphisms with splittings.*

PROOF. Immediate from 4.3 and the existence of resplendent models. □

5. Multiplicative groups of fields

Let $G = (G, \cdot, ^{-1}, 1)$ be an abelian group, written in multiplicative notation. Let $G \cup \{0\}$ be the structure G with a zero element adjoined. To be definite, $G \cup \{0\}$ is the structure $(G \cup \{0\}, \cdot, ^{-1}, 1, 0)$ with $0 \cdot x = x \cdot 0 = 0$, for all x, and $0^{-1} = 0$, or 0^{-1} undefined, if you don't mind partial functions. We identify G with $G \cup \{0\}$. Fuchs [8], problem 69, asks for a necessary and sufficient condition for G to be the multiplicative group of some field, i.e., for there to exist some binary function $+$ on $G \cup \{0\}$ so that the expanded structure is a field. Sabbagh [14] shows there is no first order solution to this problem by showing that the multiplicative group of real numbers is elementarily equivalent to a group which is not the multiplicative part of a field. This suggests the problem of finding the axioms for those G such that some $G' \equiv G$ is the multiplicative part of a field. This is a natural for the method of straightening out Henkin quantifiers.

LEMMA 5.1. *The class of structures $G \cup \{0\}$ such that G is the multiplicative part of some field is definable by a Σ_1^1 sentence, and hence is nearly axiomatizable.*

Before beginning our axiomatization, we should point out some sentence true of all such groups which is not true of all abelian groups. For example, the sentence expressing

"there are at most n x such that $x^n = 1$"

holds in the multiplicative part of any field.

LEMMA 5.2. *An abelian group G is the multiplicative part of some field F iff G is a model of the following Henkin sentence:*

$$\left.\begin{array}{l}\forall x_1 \exists y_1\\ \quad\cdot\\ \quad\cdot\\ \quad\cdot\\ \forall x_5 \exists y_5\end{array}\right\}\left\{\begin{array}{l}\bigwedge_{i,j}(x_i = x_j \rightarrow y_i = y_j)\\ \\ (x_1 = 0 \rightarrow y_1 = 1)\\ \\ (x_2 = y_1 \rightarrow y_2 = x_1)\\ \\ ((x_2 \neq 0) \wedge (x_3 = y_1 \cdot y_2) \wedge (x_4 = x_2^{-1}) \wedge (x_5 = x_1 \cdot x_4) \rightarrow (y_3 = x_2 \cdot y_3)).\end{array}\right.$$

PROOF. If you write out the Skolem form of this sentence, you notice that all five functions are equal, say to g, by the first line. The next three lines express, respectively,

$$g(0) = 1$$

$$g(g(x)) = x, \quad \text{for all } x$$

$$g(g(x) \cdot g(y)) = y \cdot g(x \cdot g(y^{-1})), \quad \text{for all} \quad x \quad \text{and all} \quad y \neq 0.$$

If G is the multiplicative part of some field, then $g(x) = 1 - x$ satisfies all the above. On the other hand, Dicker [6] shows that if $G \cup \{0\}$ has a g satisfying the above, then G is the multiplicative part of a field. □

We could have written out the natural Henkin sentence expressing the existence of a binary function + which is an Abelian group and distributes properly with respect to ·, but this is a much more complicated sentence.

DEFINITION. Let *Fld* be the set of first order approximations to the Henkin sentence given in 5.2, plus the axioms for abelian groups.

By now there is no point in writing Fld out more explicitly, especially since it is largely unintelligible. Still, it does solve the problem.

THEOREM 5.3. *Let G be a resplendent group. Then G is the multiplicative part of a field F iff G is a model of* Fld.

COROLLARY 5.4. Fld *axiomatizes the first order properties of those G which are the multiplicative parts of fields.*

COROLLARY 5.5. *If G is finite, or if G is countable and recursively saturated, then G is the multiplicative part of a field iff* $G \models$ Fld.

PROOF. In either case G is resplendent, by [2], so the result follows from 5.3. □

Problem. What are all the resplendent abelian groups?

6. The first order properties of pairs of cardinals

The examples in the previous sections were chosen for their simplicity and relevance to genuine mathematical problems. In this section we present the archetypal example of a nearly axiomatizable class because it will serve as a good introduction to the problem we discuss in [1].

We assume that the basic language L has a unary symbol U. An L structure $\mathfrak{M} = \langle M, U, \cdots\rangle$ is a *two cardinal model* if $\mathrm{Card}(M) > \mathrm{Card}(U) \geqq \aleph_0$. Vaught [17] proved that any theory S with a two cardinal model has one with $\mathrm{Card}(M) = \aleph_1$ and $\mathrm{Card}(U) = \aleph_0$. His proof also gave the following:

LEMMA 6.1. *The class of two cardinal models is nearly axiomatizable.*

PROOF. Vaught's proof shows that S has a two cardinal model iff S has a model $\mathfrak{M}$ with an $\mathfrak{M}_0 < \mathfrak{M}$, $\mathfrak{M}_0 \neq \mathfrak{M}$ and $U^{\mathfrak{M}_0} = U^{\mathfrak{M}}$. It is easy to write this down as a recursive theory in an expansion L' of L with a new unary symbol M_0. The result follows from 1.1. The real trick to axiomatizing this class, though, as noticed by Keisler, is to use a stronger condition. Vaught's proof also shows that S has a two cardinal model iff S has a model $\mathfrak{M}$ with an $\mathfrak{M}_0$ as above but with $\mathfrak{M} \cong \mathfrak{M}_0$. □

Knowing 6.1, Vaught asked for an explicit set of axioms for two cardinal models. These were given by Keisler [12]. We show how to get a similar set of axioms by straightening Henkin formulas.

LEMMA 6.2. *Let $\mathfrak{M} = \langle M, U, \cdots\rangle$ be a resplendent L-structure. The following are equivalent:*

i) *$\mathfrak{M}$ is isomorphic to some proper elementary submodel $\mathfrak{M}_0 = \langle M_0, U, \cdots\rangle$ with the same U.*

ii) *For each finite set s of formulas of L, M is a model of $\exists z H_s(z)$, where $H_s(z)$ is the following Henkin formula ($s = s(x_1 \cdots x_n)$ is assumed to have its free variables among $x_1 \cdots x_n$):*

$$\left.\begin{matrix}\forall x_1\exists y_1\\ \cdot\\ \cdot\\ \cdot\\ \forall x_n\exists y_n\\ \forall u\exists v\end{matrix}\right\}\quad\left\{\begin{matrix}\bigwedge_{i,j}(x_i = x_j \to y_i = y_j)\\ \\ (y_1 \neq z)\\ (U(u)\wedge x_1 = v \to y_1 = u)\\ \bigwedge_{\psi\in s}(\psi(x_1\cdots x_n)\to \psi(y_1\cdots y_n)).\end{matrix}\right.$$

PROOF. To prove (i) $\Rightarrow$ (ii), let $\mathfrak{M}_0$ be a proper elementary submodel of $\mathfrak{M}$ with the same U and let $f: \mathfrak{M} \cong \mathfrak{M}_0$. Let $z \in \mathfrak{M} - \mathfrak{M}_0$. To show that $\mathfrak{M}$ is a model of $H_s(z)$, choose y_i as $f(x_i)$ and, for $u \in U$, choose $v = f^{-1}(u)$. To prove (ii) $\Rightarrow$ (i), we use Theorem 2.8. By that result, there is a $z \in M$ such that $\mathfrak{M}$ is a model of:

$$\left.\begin{array}{c} \forall u \exists v \\ \forall x_1 \exists y_1 \\ \cdot \\ \cdot \\ \cdot \\ \forall x_n \exists y_n \\ \cdot \\ \cdot \\ \cdot \end{array}\right\} \left\{\begin{array}{l} \bigwedge_{i,j}(x_i = x_j \rightarrow y_i = y_j) \\ (y_1 \neq z) \\ \\ (U(u) \wedge x_1 = v \rightarrow y_1 = u) \\ \\ \bigwedge_{\psi \in L}(\psi(x_1 \cdots x_n) \rightarrow \psi(y_1 \cdots y_n)). \end{array}\right.$$

This asserts the existence of two functions f, g such that f is an elementary embedding of $\mathfrak{M}$ into itself with $z \notin \text{range}(f)$, and such that for all $u \in U$, $f(g(u)) = u$. Thus U is in the range of f so if we let $\mathfrak{M}_0$ be the range of f, we have (i). □

DEFINITION. Let *Vau* be the first order theory whose axioms are all prenex formulas beginning with a quantifier string

$$\exists z \forall x_1 u_1 \exists y_1 v_1 \cdots \forall x_n u_n \exists y_n v_n$$

followed by a conjunction of formulas of the following forms:

$$(x_i = x_j \rightarrow y_i = y_j)$$

$$(u_i = u_j \rightarrow v_i = v_j)$$

$$(U(u_i) \wedge x_i = v_i \rightarrow y_i = u_i)$$

$$(y_i \neq z)$$

$$(\psi(x_1 \cdots x_n) \rightarrow \psi(y_1 \cdots y_n)).$$

This is not exactly the set of axioms arrived at in Keisler [12], but they are clearly equivalent.

THEOREM 6.3. *Let* $\mathfrak{M} = \langle M, U, \cdots \rangle$ *be a resplendent structure. The following are equivalent:*

i) $\mathfrak{M}$ *is elementarily equivalent to a two cardinal model.*

ii) $\mathfrak{M} \models \text{Vau}$.

iii) $\mathfrak{M}$ *is isomorphic to a proper elementary submodel of itself with the same* U.

iv) $\mathfrak{M}$ *has a proper elementary submodel with the same* U.

PROOF. The main implication, given Vaught's work, is (ii) ⇒ (iii). So assume (ii). As in the previous results (3.3, 4.3) it is easy to see that each formula of the form $\exists z\psi(s)$, for ψ in some $\text{App}_{H_s}(z)$, is implied by some formula in Vau. This just amounts to permuting quantifiers. Thus, by 6.2, (iii) holds.

To prove (iii) ⇒ (ii), let f be the elementary embedding, $z \notin \text{rng}(f)$. Let $y_i = f(x_i)$ and, for $u_i \in U$, let $v_i = f^{-1}(u_i)$. This shows that $\mathfrak{M} \models \text{Vau}$.

The implication (iii) ⇒ (iv) is trivial. The implication (iv) ⇒ (i) follows from the result of Vaught mentioned in the proof of 6.1. This leaves us with the proof of (i) ⇒ (iii). Let T'' be the theory in an expansion of L with unary M_0 and unary f expressing that $\mathfrak{M}_0 < \mathfrak{M}$ and $f: \mathfrak{M} \cong \mathfrak{M}_0$. By the second result of Vaught used in 6.1, $\text{Th}(M)$ is consistent with T''. Since $\mathfrak{M}$ is resplendent, some expansion of $\mathfrak{M}$ is a model of T'', by 2.4(v) in [2]. Thus (iii) holds. □

COROLLARY 6.4. Vau *is a set of axioms for two cardinal models.*

PROOF. Immediate from 6.3. □

Gregory, in a recent paper in the JSL, has found a much better set of axioms, but he has to work much harder, too.

COROLLARY 6.5. *Let* $\mathfrak{M}$ *be a countable, recursively saturated model of* Vau. *There is an elementary extension* $\mathfrak{M}'$ *of* $\mathfrak{M}$ *of power* $\aleph_1$ *such that* $U^M = U^{M'}$.

PROOF By [2], $\mathfrak{M}$ is resplendent and homogeneous. By 6.3, $\mathfrak{M}$ is isomorphic to a proper elementary extension of itself with the same U. By the usual proof of Vaught's Theorem, as given in Chang-Keisler [3], for example, we can build $\mathfrak{M}'$ as the union of an ω_1-chain of structures each isomorphic to $\mathfrak{M}$. □

7. Game sentences and other concluding remarks

Once we have the set of axioms for a nearly axiomatizable class $\mathcal{K}$, it is usually easy to see that they do indeed provide a set of axioms, without ever considering Henkin formulas. We have deliberately presented our examples in a way which emphasizes the way Henkin formulas help us *find* the axioms in question. The only problem that can arise, as illustrated in section 6, is that it may not be at all obvious how to go from a particular T' (as in Lemma 1.1) to a Henkin formula.

The idea of approximating Henkin formulas was suggested to us by some analogous results of Keisler [10] on approximating certain types of infinitary game formulas by means of finite expressions, on saturated models. (Keisler used his approximations to prove certain preservation theorems, in [11]. It is not difficult to see how to prove his results via resplendent models and Henkin formulas, in a similar way.) The *game formula* (more accurately recursive closed game formula) associated with the Henkin formula $H(z)$

$$\left.\begin{matrix} \forall x_1 \exists y_1 \\ \cdot \\ \cdot \\ \cdot \\ \forall x_n \exists y_n \end{matrix}\right\} \phi(z, x_1, y_1, \cdots, x_n, y_n)$$

is the natural limit of the finite approximations defined in section 2. That is, it is the "formula" $\mathcal{G}(z)$ with infinite quantifier prefix

$$\forall x^1 \exists y^1 \cdots \forall x \exists y^k \cdots$$

and whose matrix consists of the conjunction of all $\phi(z, \vec{x}^i y^i)$ and all $(x^i_i = x^{i'}_i \rightarrow y^i_i = y^{i'}_i)$. The proof of 2.4 shows that on countable structures $\mathfrak{M}$, $\mathfrak{M} \models H(z)$ iff $\mathfrak{M} \models \mathcal{G}(z)$. Combining this with Theorem 2.2, gives a simple new, and much more explicit, proof of Svenonius' Theorem to the effect that every Σ^1_1 formula is equivalent, on countable structures, to a recursive closed game formula. (See theor. 6.8 of [0].) Our original proof of Theorem 2.4 went by means of proving this directly and then using exercise 7.17 (iii) of [0]. It seemed more appropriate, in a talk dedicated to Abraham Robinson, to deal with finitary first order logic, without the detour through game formulas. Makkai suggested to us that this new explicit form for Svenonius' Theorem is important in its own right, and should be pointed out here.

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